Development and Control of a Low-Cost Cart-Pendulum Educational Platform

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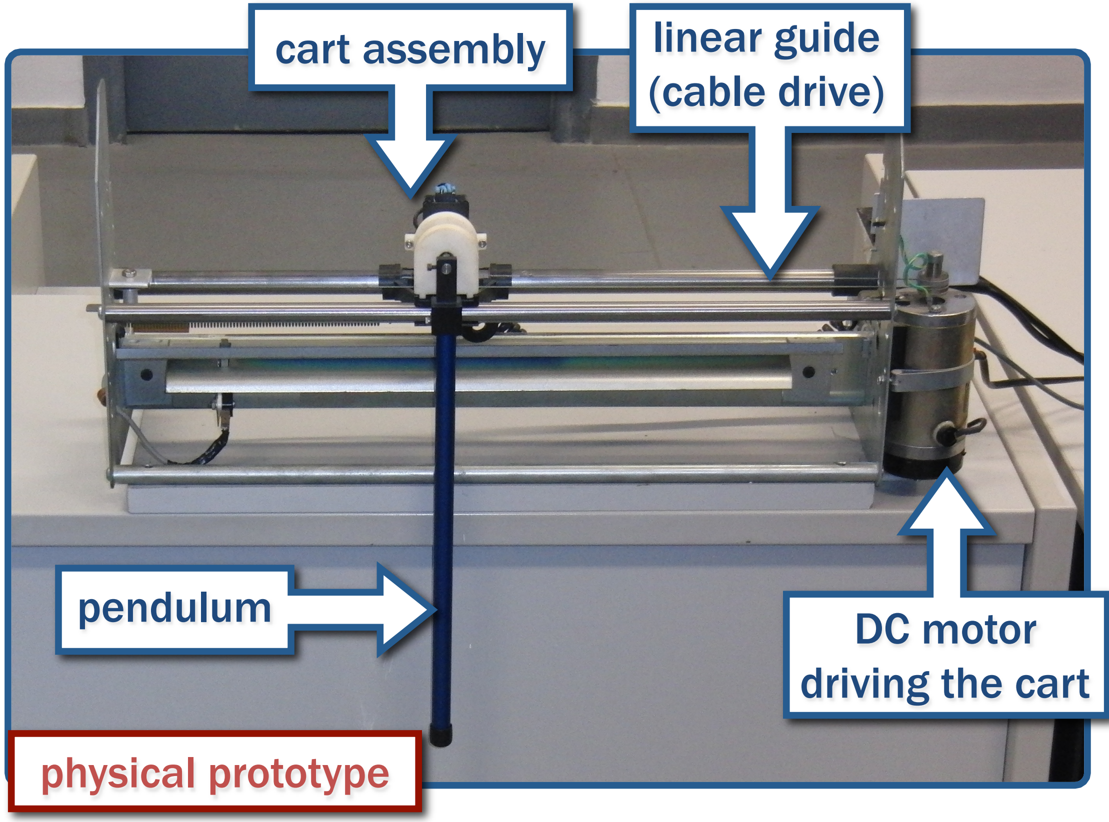
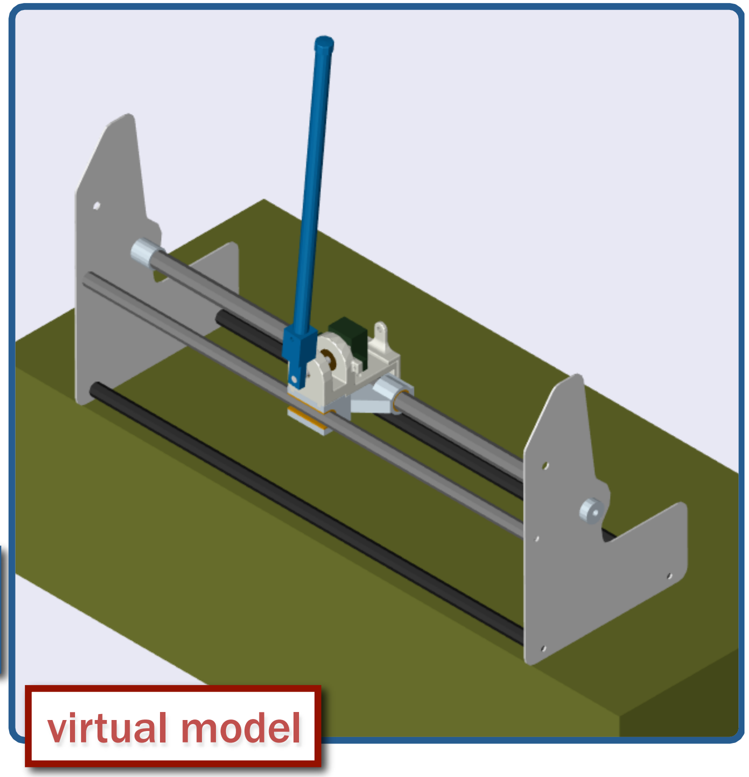
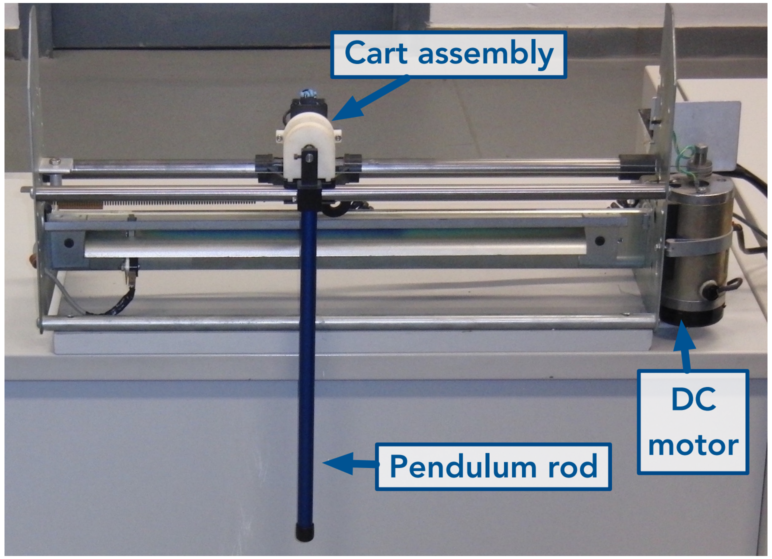
Cart-Pendulum Platform

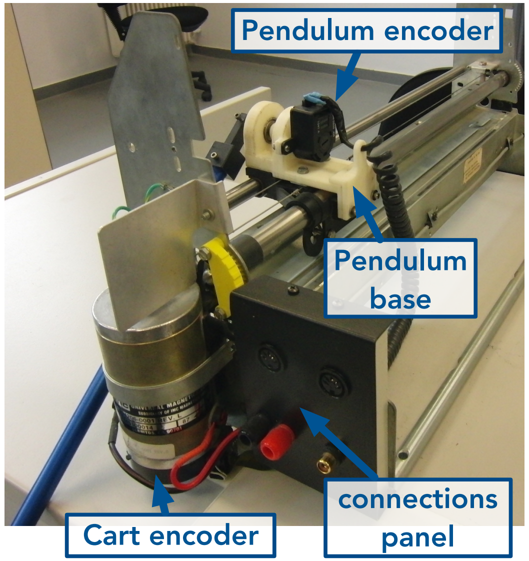




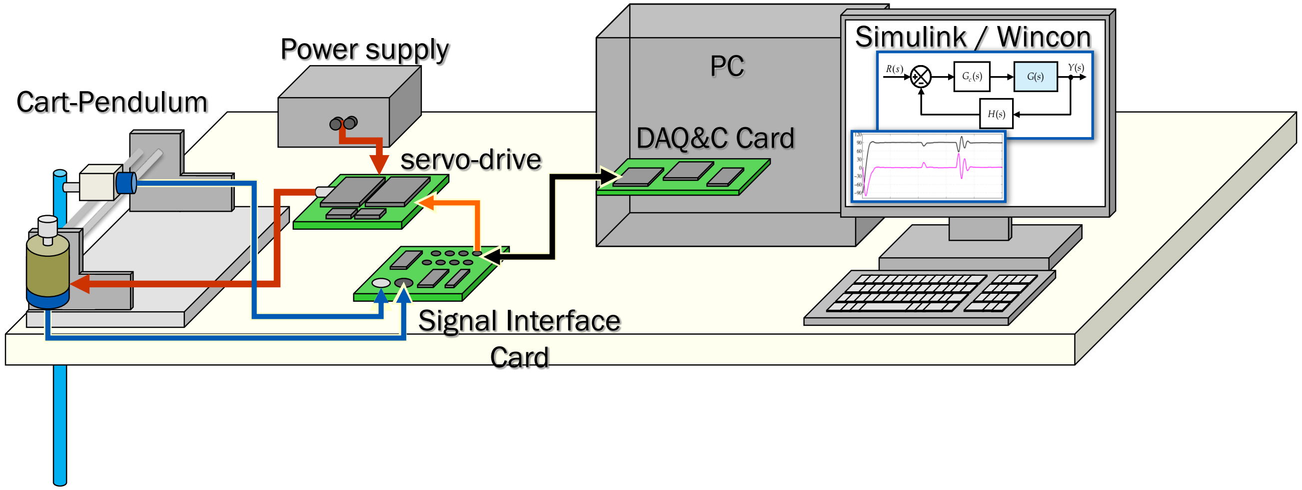
1. Describe Cart-Pendulum Platform
2. Nonlinear dynamic equations of motion
3. Generic model of nonlinear friction
4. Identification of friction parameters
5. Linearisation of dynamic equations, State-space models, State-feedback Controllers
6. Pole placement design
7. Linear Quadratic Regulator (LQR) design

Overall architecture



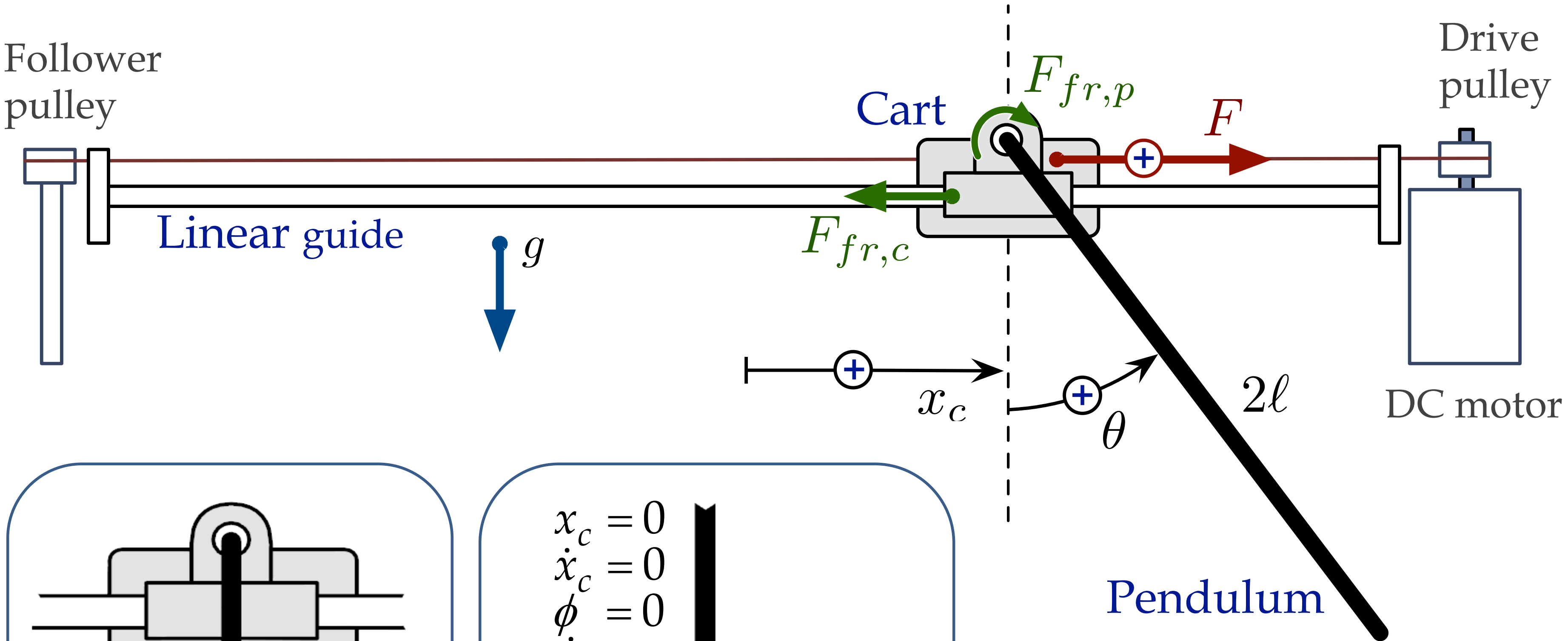


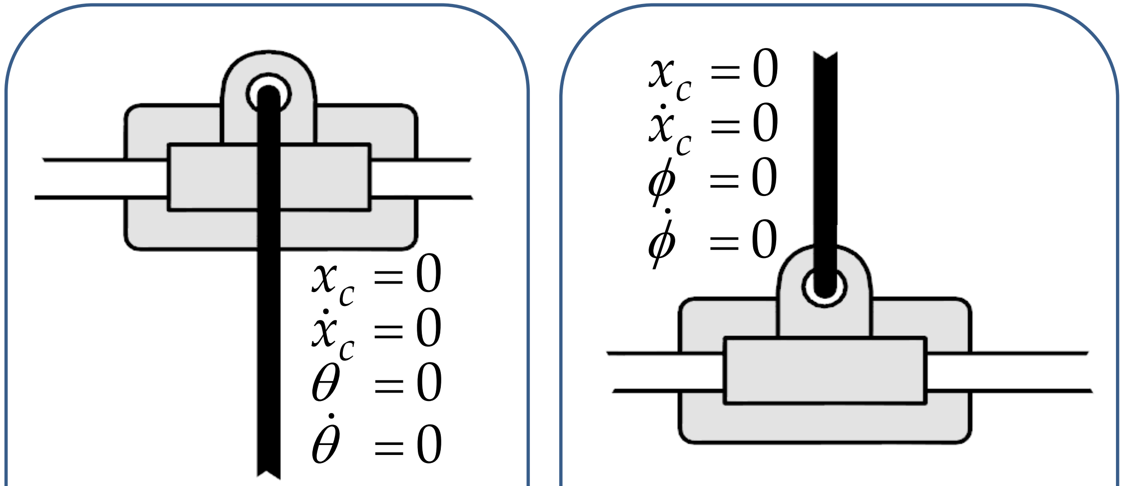
Details of the cart-pendulum platform



Overall architecture of the developed cart-pendulum platform

Cart-Pendulum Schematic





Stable equilibrium point (gantry crane)

Unstable equilibrium point (inverted pendulum)

- $F_{fr,c}$ ⇒ pendulum's friction
- $F_{fr,c}$ ⇒ cart's friction
- generalized coordinates: $q = \begin{bmatrix} x_c & \theta \end{bmatrix}^{\mathrm{T}}$
- F ⇒ Control input in $[N]$:
  $F = \propto v_a$, where
  v_a ⇒ DC motor voltage

Nonlinear Dynamic Equations of Motion

$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{V}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{G}(\boldsymbol{q}) + \boldsymbol{F_{fr}}(\dot{\boldsymbol{q}}) = \boldsymbol{u}$ Lagrangian and Dynamics

- $\boldsymbol{M}(\boldsymbol{q})$: Inertia matrix
- $\ddot{\boldsymbol{q}}$: Acceleration vector
- $\boldsymbol{V}(\boldsymbol{q}, \dot{\boldsymbol{q}})$: Centripetal vector
- $\boldsymbol{G}(\boldsymbol{q})$: Gravity vector
- $\boldsymbol{F_{fr}}(\dot{\boldsymbol{q}})$: Nonlinear friction vector
- $\boldsymbol{u}$: Generalized force vector

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} m_1/\left(I_p + m_p\ell^2\right) & 0 \\ 0 & m_1/(m_p\cos\theta) \end{bmatrix}, \quad \ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{bmatrix}, \quad \boldsymbol{V}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} m_p\ell\dot{\theta}^2\sin\theta \\ -m_p\ell^2\dot{\theta}^2\sin\theta \end{bmatrix}$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \left(gm_p^2\ell^2\sin(2\theta)\right)/\left(2\left(I_p + m_p\ell^2\right)\right) \\ -g\left(m_c + m_p\right)\ell\tan\theta \end{bmatrix}$$

$$\boldsymbol{F_{fr}}(\dot{\boldsymbol{q}}) = \begin{bmatrix} -F_{fr,c}(\dot{x}_c) + \left(m_p\ell\cos\theta F_{fr,p}(\dot{\theta})\right)/\left(I_p + m_p\ell^2\right) \\ \ell F_{fr,c}(\dot{x}_c) - \left(\left(m_c + m_p\right)F_{fr,p}(\dot{\theta})\right)/(m_p\cos\theta) \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} F \\ -\ell F \end{bmatrix}$$

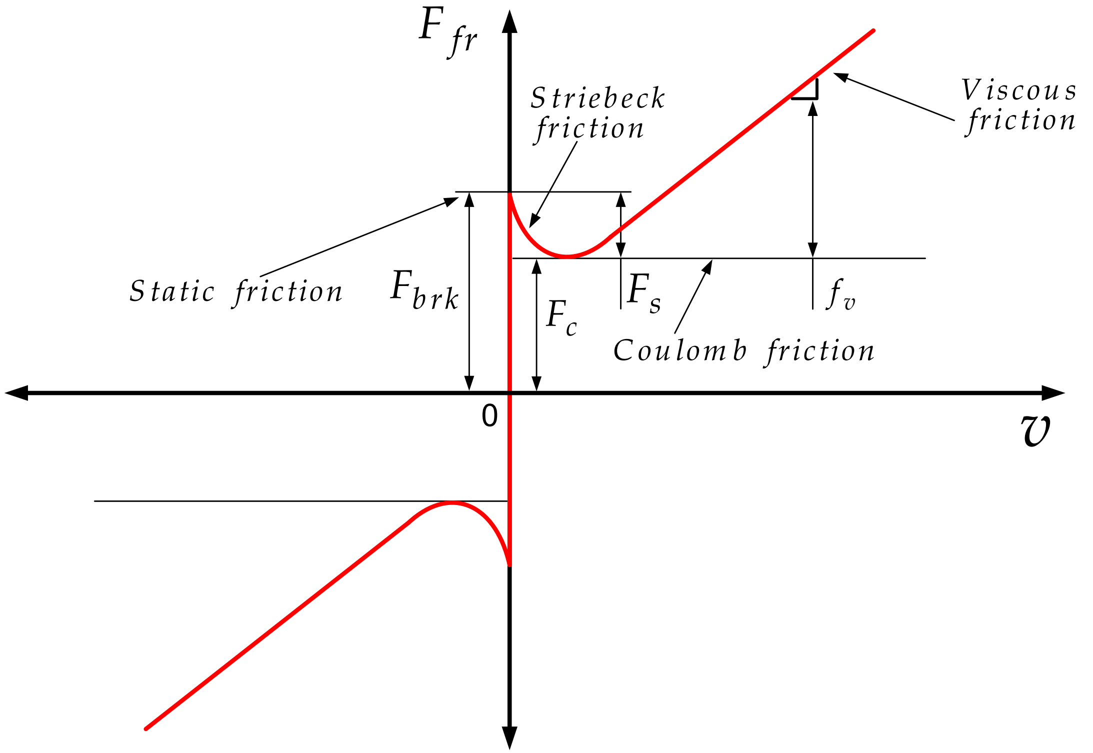
where, $m_1 = \left(m_c + m_p\right)I_p + m_c m_p\ell^2 + m_p^2\ell^2\sin^2\theta$

measured or calculated paremeters: m_c, m_p, I_p, ℓ, g

Generic model of nonlinear friction

$$\text{Nonlinear friction model :} \begin{cases} F_{fr,c}(\dot{x}_c) \Rightarrow \text{nonlinear cart friction} \\ \Uparrow \\ F_{fr}(v) = \left(F_c + (F_{brk} - F_c)\, e^{(-c_v|v|)}\right) sign\,(v) + f_v v \\ \Downarrow \\ F_{fr,p}(\dot{\theta}) \Rightarrow \text{nonlinear pendulum friction} \end{cases}$$

1. $F_{brk} \Longrightarrow$ Static friction
2. $F_c \Longrightarrow$ Coulomb friction
3. $f_v \Longrightarrow$ viscous friction coefficient
4. $c_v \Longrightarrow$ Stribeck velocity coefficient
5. $e^{(-c_v|v|)} \Longrightarrow$ Stribeck friction



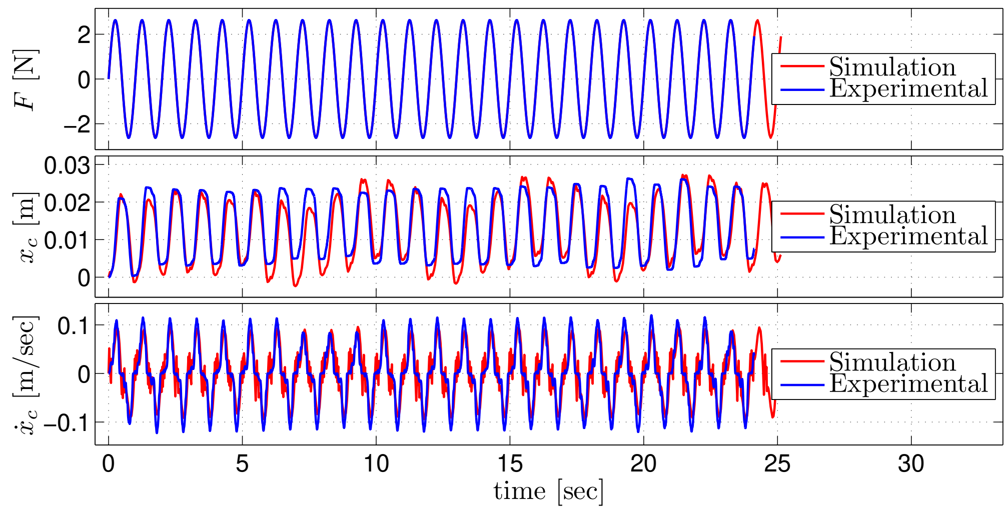
Identification of friction parameters

1. Empirical identification of cart's nonlinear friction parameters
   - sinusoidal applied force:
     $F \Rightarrow$ Control input in $[N]$:
     $F = \propto v_a$, where
     $v_a \Rightarrow$ DC motor voltage
   - record x_c and $\dot{x}_c$
   - identify cart's friction parameters with $\ddot{x}_c$ nonlinear dynamic equation
2. Empirical identification of pendulum's nonlinear friction parameters
   - record free response of the pendulum
   - identify pendulum's friction parameters with $\ddot{\theta}$ nonlinear dynamic equation

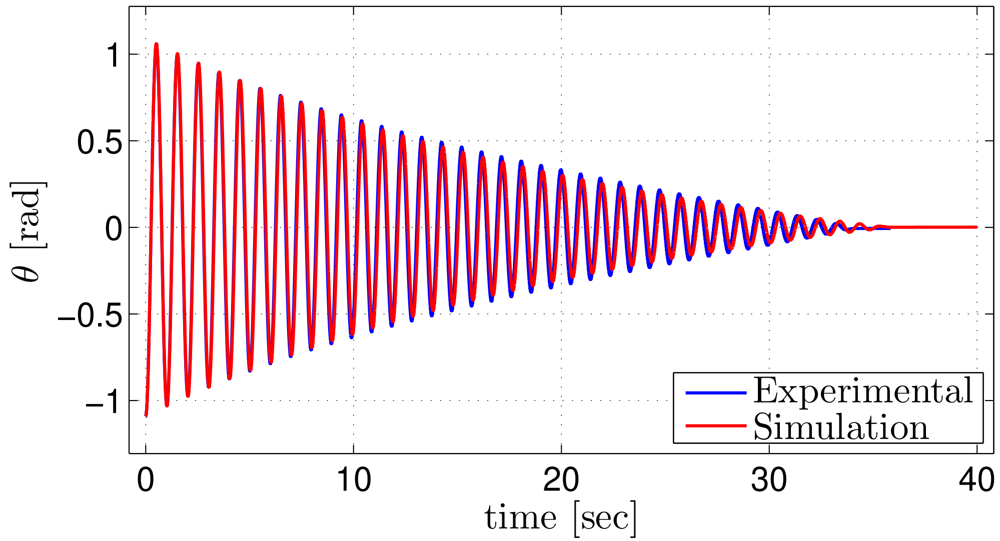
Friction parameters:

- $F_{brk} \Longrightarrow$ Static friction
- $F_c \Longrightarrow$ Coulomb friction
- $f_v \Longrightarrow$ viscous friction coefficient
- $c_v \Longrightarrow$ Stribeck velocity coefficient

Results of fitting cart's nonlinear friction parameters

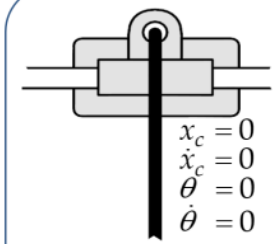


Results of fitting pendulum's nonlinear friction parameters

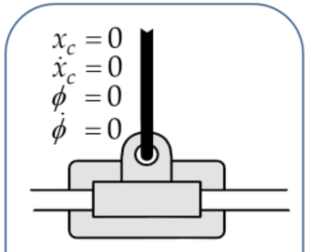


Linearisation, State-Space Models, State-feedback

Nonlinear Dynamic Equations



Stable equilibrium point (gantry crane)



Unstable equilibrium point (inverted pendulum)

Jacobian Linearisation

$$\phi = \theta - \pi$$

$$\boldsymbol{x} = \begin{bmatrix} x_c & \dot{x}_c & \theta & \dot{\theta} \end{bmatrix}^{\mathrm{T}} \Leftarrow \text{state vectors} \Rightarrow \boldsymbol{x} = \begin{bmatrix} x_c & \dot{x}_c & \phi & \dot{\phi} \end{bmatrix}^{\mathrm{T}}$$

State-Space Models

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_G x + \boldsymbol{B}_G u \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_I x + \boldsymbol{B}_I u \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

Augmented State-Space Models

$$\text{new state variable } \rightarrow w \Rightarrow \boldsymbol{x}' = \begin{bmatrix} x_c & \dot{x}_c & \theta & \dot{\theta} & w \end{bmatrix}^{\mathrm{T}}$$

$$w = \int_0^t (r - x_c)\, d\tau \text{ or, equivalently } \dot{w} = r - x_c$$

$$\dot{\boldsymbol{x}}' = \boldsymbol{A}'_G \boldsymbol{x}' + \boldsymbol{B}'_G u + \boldsymbol{H}r \qquad \boldsymbol{y}' = \boldsymbol{C}'\boldsymbol{x}'$$

$$\dot{\boldsymbol{x}}' = \boldsymbol{A}'_I \boldsymbol{x}' + \boldsymbol{B}'_I u + \boldsymbol{H}r \qquad \boldsymbol{y}' = \boldsymbol{C}'\boldsymbol{x}'$$

State-feedback Controllers

Specify control signal $u = F$ as $u = -\boldsymbol{K}\boldsymbol{x}'$

Pole Placement

$$\dot{\boldsymbol{x}}' = \left(\boldsymbol{A}'_G - \boldsymbol{B}'_G \boldsymbol{K}\right)\boldsymbol{x}' + \boldsymbol{H}r$$

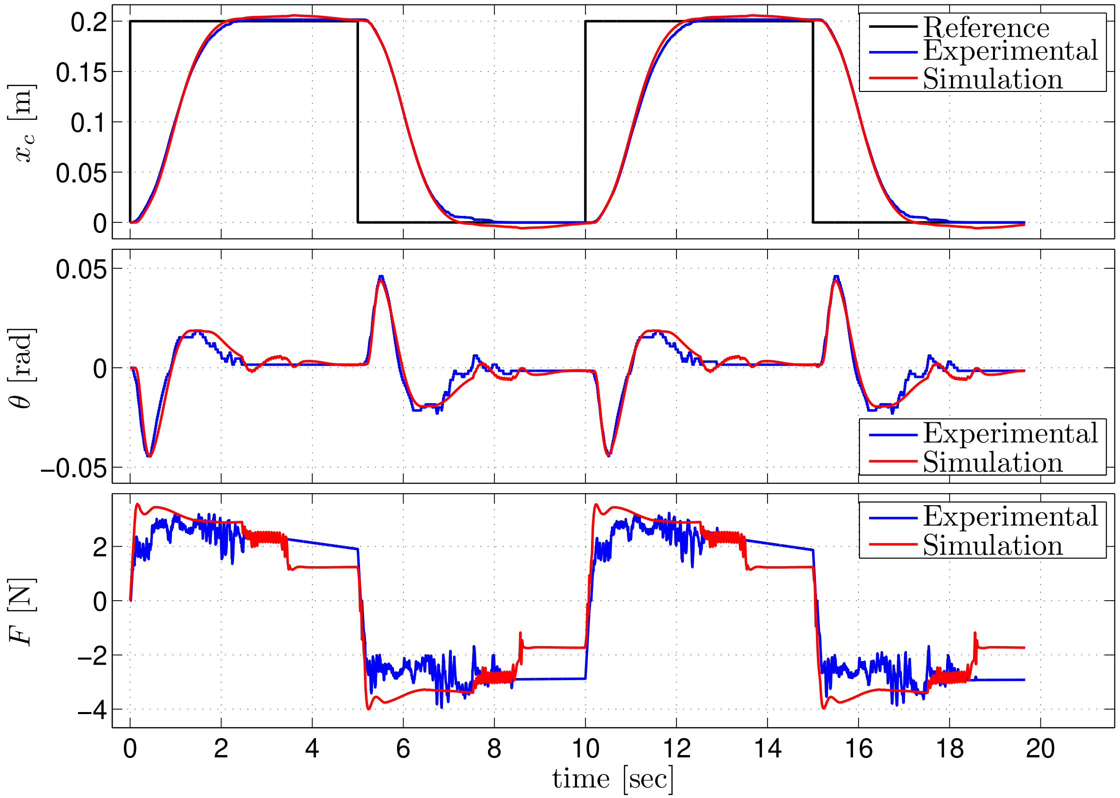
Linear Quadratic Requlator (LQR)

$$\dot{\boldsymbol{x}}' = \left(\boldsymbol{A}'_I - \boldsymbol{B}'_I \boldsymbol{K}\right)\boldsymbol{x}' + \boldsymbol{H}r$$

Pole placement design - gantry crane

- Closed loop feedback system matrix is $\boldsymbol{A'}_{\boldsymbol{G}} - \boldsymbol{B'}_{\boldsymbol{G}}\boldsymbol{K}$ and $\boldsymbol{A'}_{\boldsymbol{G}} \in R^{n\times n}$, where $n = 5$.
- Desired 5th order characteristic polynomial will be of the following format:

$$(s - p_3)(s - p_4)(s - p_5)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)$$



design specifications

- settling time $t_s = 2.0$ s
- maximum overshoot 2%

$\Downarrow$

- damping coefficient $\zeta = 0.78$
- natural frequency $\omega_n = 2.565$ rad / s
- dominant complex poles, i.e.:

$$p_{1,2} = -\zeta\omega_n \pm \left(\omega_n\sqrt{1-\zeta^2}\right) i$$

- choose $p_j >> -\zeta\omega_n$, $j = 3, 4, 5$
- $\{-6, -13, -14, -2 \pm 1.6i\}$ of the feedback system matrix $\boldsymbol{A'}_{\boldsymbol{G}} - \boldsymbol{B'}_{\boldsymbol{G}}\boldsymbol{K}$, yielded the following gain vector $\boldsymbol{K}$:

$$\boldsymbol{K} = \begin{bmatrix} 134.67 & 45.15 & -55.60 & 4.09 & -146.18 \end{bmatrix}$$

Pole placement implementation - gantry crane

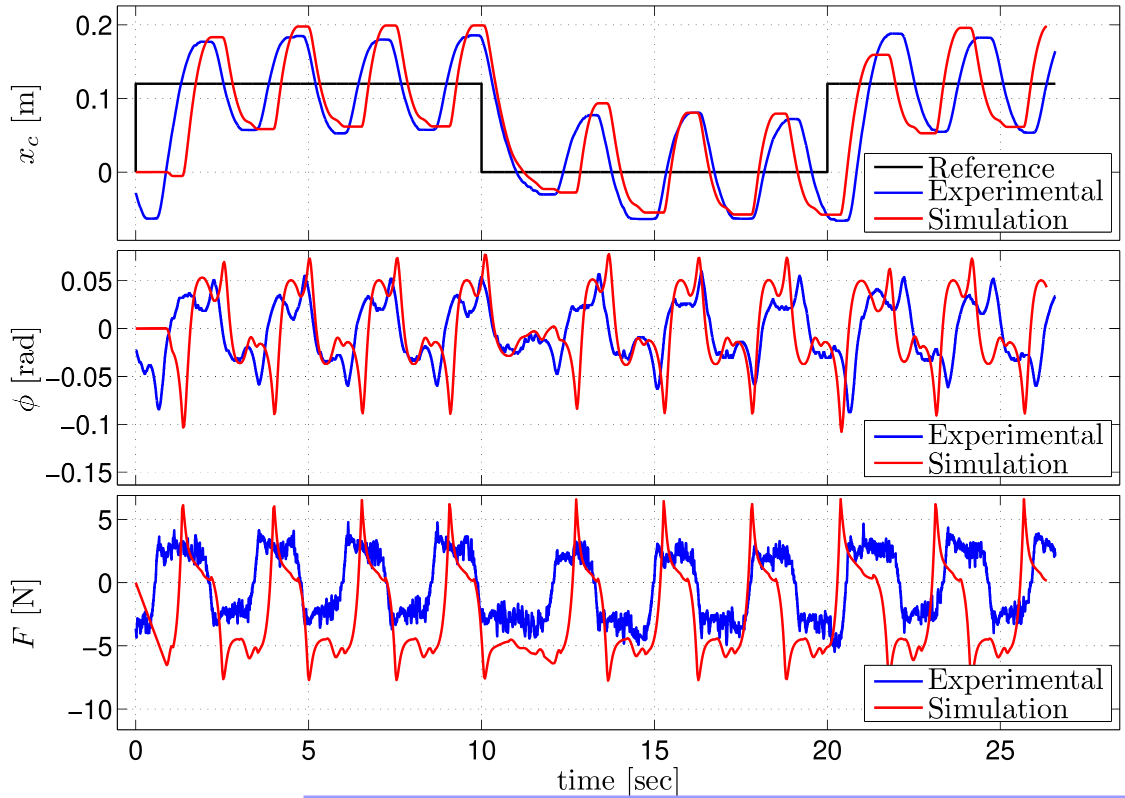
1. Open Loop response experiment
2. Reference $r = x_c(0)$, disturbance rejection experiment
3. trajectory tracking, full state-feedback with integrator (augmented)

Linear quadratic regulator (LQR) design - inverted pendulum

- The LQR design yields an optimal controller that minimizes the following infinite horizon quadratic cost function:

$$J = \int_0^\infty \left(\boldsymbol{x'}^{\mathrm{T}} \boldsymbol{Q} \boldsymbol{x'} + \boldsymbol{u}^{\mathrm{T}} \boldsymbol{R} \boldsymbol{u} \right) dt \ \text{ where, } \ u = -\boldsymbol{K}\boldsymbol{x'}$$

- choose weight matrices $\boldsymbol{Q} = \mathrm{diag}\{200, 0, 500, 0, 1300\}$ and $\boldsymbol{R} = [0.35]$.



- optimized gain values are derived from: $\boldsymbol{K} = \boldsymbol{R}^{-1} \boldsymbol{B'}_{\boldsymbol{I}}{}^{\mathrm{T}} \boldsymbol{P}$

- where $\boldsymbol{P}$ is found by the solution of the Riccati algebraic equation:

$$\boldsymbol{A'}_{\boldsymbol{I}}{}^{\mathrm{T}} \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A'}_{\boldsymbol{I}} - \boldsymbol{P} \boldsymbol{B'}_{\boldsymbol{I}} \boldsymbol{R}^{-1} \boldsymbol{B'}_{\boldsymbol{I}}{}^{\mathrm{T}} \boldsymbol{P} + \boldsymbol{Q} = 0$$

- eigenvalues of the closed loop system matrix $\boldsymbol{A'}_{\boldsymbol{I}} - \boldsymbol{B'}_{\boldsymbol{I}} \boldsymbol{K}$ are then found to be $\{-11.7 \pm 8.3i, -1.8 \pm 1.9i, -2.1\}$

$$\boldsymbol{K} = \begin{bmatrix} -67.25 & -38.33 & 95.73 & 13.07 & 60.94 \end{bmatrix}$$

Linear quadratic regulator (LQR) implementation - inverted pendulum

Virtual model of the cart-pendulum system

Thank you!

CSRL

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